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REFERENCES

1. R. D. CESS, The interaction of thermal radiation with conduction and convection heat transfer, *Advances in Heat Transfer*, Vol. 1, pp. 1–50. Academic Press, New York (1964).
2. R. VISKANTA, Radiation transfer and interaction of convection with radiation heat transfer, *Advances in Heat Transfer*, Vol. 3, pp. 175–251. Academic Press, New York (1966).
3. E. M. SPARROW and R. D. CESS, *Radiation Heat Transfer*. Brooks/Cole, Belmont, California (1966).
4. W. G. VINCENTI and C. H. KRUGER, *Introduction to Physical Gas Dynamics*. John Wiley, New York (1965).
5. W. SCHNEIDER, Grundlagen der Strahlungsgasdynamik, *Acta Mechanica* **5**, 85–117 (1968).
6. D. B. OLFE, A modification of the differential approximation for radiative transfer, *AIAA Jl* **5**, 638–643 (1967).
7. L. R. GLICKSMAN, An approximate method for multidimensional problems of radiative energy transfer in an absorbing and emitting media, *J. Heat Transfer* **9:C**, 502–510 (1969).
8. M. M. R. WILLIAMS, The temperature distribution in a radiating fluid flowing over a flat plate, *Q.J. Mech. Appl. Math.* **22**, 487–500 (1969).
9. Y. TAITEL, Exact solution for the "radiation layer" over a flat plate, *J. Heat Transfer* **91C**, 188–189 (1969).
10. M. ABRAMOWITZ and I. A. STEGUN (editors), *Handbook of Mathematical Functions*. Dover Publications, New York, p. 228 ff. (1965).
11. V. S. ARPACI, Unsteady radiation slip, *AIAA Jl* **8**, 1910–1913 (1970).
12. A. ERDELYI (ed.), *Tables of Integral Transforms*, Vol. 1. McGraw Hill, New York (1954).
13. I. S. GRADSHTEYN and I. M. RYZHIK, *Tables of Integrals, Series and Products*. Academic Press, New York (1965).

Int. J. Heat Mass Transfer. Vol. 15, pp. 2667–2670. Pergamon Press 1972. Printed in Great Britain

ON HEAT TRANSFER IN MHD CHANNEL FLOW

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NOMENCLATURE

a, half width of the channel;

B, rheological parameter of the Prandtl–Eyring model;

c, specific heat;

C, rheological parameter of the Prandtl–Eyring model;

E, electric field strength;

F, incomplete elliptic integral of the first kind;

${}_3F_2$, generalized second order hypergeometric function (Clausen function);

H, magnetic field strength;

Ha, $\sqrt{(SRe)}$, Hartmann number;

j, electric current density;

k, thermal conductivity;

$k_{(1)}$, $\sqrt{\{1-[Ha_2/\tau_2^{*\prime}(0)]^2\}}$, $k_{(2)} = \sqrt{\{1-[\tau_2^{*\prime}(0)/Ha_2]^2\}}$, modulus of Jacobian elliptic functions and integrals;

K, $-\dfrac{E_z}{\mu_e H_y u_0}$, external loading parameter;

Kz, $\dfrac{aC}{u_0}$, characteristic parameter;

n, power law exponent;

K^*, $K - \dfrac{1}{S}\dfrac{\partial p^*}{\partial x^*}$, parameter containing pressure gradient, Stuart number and external loading parameter;

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p, pressure;
Pe, $\frac{ca\rho u_0}{k}$, Péclét number;
Pr, Pe/Re, Prandtl number;
Re_1, $\rho u_0^{(2n-1)/n} a^{1/n}/\mu_p^{1/n}$, $Re_2 = \frac{\rho a u_0 C}{B}$, Reynolds numbers;
S, $\frac{\sigma_e a \mu_e^2 H_y^2}{\rho u_0}$, Stuart number;
T, temperature;
u, velocity in x-direction;
x, y, z, space coordinates.

Greek symbols
θ, $T - T_w$, temperature difference;
τ, shear stress;
μ_p, rheological parameter of the power law;
ρ, density;
μ_e, magnetic permeability;
σ_e, electrical conductivity;
$\varphi_{(1)}$, $\sin^{-1}(\tanh \tau_2^*)$, $\varphi_{(2)} = \sin^{-1}(1/\cosh \tau_2^*)$, arguments of elliptic integrals.

Subscripts
0, at the center axis, $y = 0$;
1, power law fluid;
2, Prandtl–Eyring fluid;
w, wall;
∞, thermally fully developed;
y, z, in y-, z-direction.

Superscripts
$*$, dimensionless quantity;
$'$, $\mathrm{d}(\ )/\mathrm{d}y^*$.

INTRODUCTION

In the past several years, there have been many investigations on the convective heat transfer in the thermal entrance region for the steady-state, constant-property, laminar MHD channel flow of Newtonian fluids [1–7]. The cases of prescribed constant wall temperature and constant wall heat flux have been solved by different methods and assumptions. Following the procedure commonly used for solving this problem [6], the general solution of the linear energy equation can be found in two parts which are combined to give the complete temperature distribution. One part is the solution of the thermally fully developed limiting case, which was analytically obtained for MHD channel flow of Newtonian fluids [8] by integrating the energy equation twice.

MHD channel flow of electrically conducting power law fluids and Prandtl–Eyring fluids has been investigated by Martinson and Pavlov [9] and Schroeder [10]. In both cases, velocity and shear stress cannot be expressed explicitly in terms of the distance across channel height. Therefore, considering fully developed MHD channel flow of power law fluids, Yang and Ou [11] solved numerically the energy equation for the thermally fully developed limiting case.

The purpose of this note is to show that the temperature in the thermally fully developed region can be expressed in terms of the shear stress and the distance across channel height. Therefore, temperature distributions for the fully developed MHD channel flow of non-Newtonian fluids can be calculated easily, if the shear stress distributions are known. Assuming constant wall temperature, the flow of power law fluids and Prandtl–Eyring fluids is considered. The case of constant wall heat flux can be treated analogously.

ANALYSIS

The geometry under consideration, illustrated in Fig. 1, consists of two infinite parallel plates extending in the x- and z-directions. The channel width is $2a$. The electrically conducting fluid flows with the velocity $u(y)$ in the x-direction, a magnetic field with the constant magnetic field strength H_y is imposed in the y-direction. There is an electric field with the constant electric field strength E_z and an electric current j_z in the z-direction. The flow is steady, laminar, incompressible and fully developed, and the physical properties of the fluid are constant. The Hall effect and the electric charge are neglected. There are no velocity components in the y- and z-directions and no magnetic field in the z-direction. All variables are independent of the z-direction.

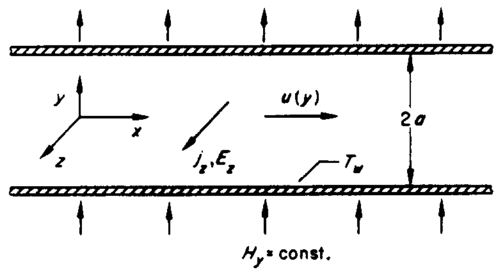

FIG. 1. Channel geometry.

For this configuration, the energy equation for the thermally fully developed limiting case with constant wall temperature conditions can be written as

$$k\frac{\mathrm{d}^2 T_\infty}{\mathrm{d}y^2} + \tau\frac{\mathrm{d}u}{\mathrm{d}y} + \sigma_e(E_z + \mu_e H_y u)^2 = 0. \qquad (1)$$

With the following dimensionless variables and parameters:

$$x^* = \frac{x}{a}; \quad y^* = \frac{y}{a}; \quad u^* = \frac{u}{u_0};$$

$$\theta^*_\infty = c\frac{T_\infty - T_w}{u_0^2}; \quad \tau^* = \frac{\tau}{\rho u_0^2}; \quad p^* = \frac{p}{\rho u_0^2}$$

the dimensionless form of the energy equation becomes

$$\frac{1}{Pe}\theta^{*\prime\prime}_\infty + \tau^* u^{*\prime} + S(u^* - K)^2 = 0. \tag{2}$$

The thermal boundary conditions are

$$\theta^*_\infty(\pm 1) = 0. \tag{3}$$

The following relation between velocity and shear stress, which was derived from the momentum equations, the generalized Ohm's law and Maxwell's equations is given in [10].

$$u^* = \frac{1}{S}\tau^{*\prime} + K^*. \tag{4}$$

Considering (4), the integration of the energy equation gives

$$\frac{1}{Pe}\theta^{*\prime}_\infty + \frac{1}{S}\tau^*\tau^{*\prime} - \frac{2}{S}\frac{\partial p^*}{\partial x^*} + \frac{1}{S}\left(\frac{\partial p^*}{\partial x^*}\right)^2 y^* = 0. \tag{5}$$

Because of the thermal boundary conditions (3) and the condition that the shear stress at the center axis $\tau^*(0) = 0$, the integration constant becomes zero. Integrating (5) gives an expression for the temperature in terms of the shear stress and the coordinate.

$$\theta^*_\infty = \frac{Pe}{2S}\left[\tau^{*2}(1) - \tau^{*2} + \left(\frac{\partial p^*}{\partial x^*}\right)^2(1 - y^{*2}) - 4\frac{\partial p^*}{\partial x^*}\int_{y^*}^{1}\tau^* \mathrm{d}y^*\right].$$

MHD CHANNEL FLOW OF POWDER LAW FLUIDS

For plane laminar channel flow of power law fluids, the dimensionless stress–strain relation can be written as

$$u^{*\prime} = Re_1|\tau^*|^{n-1}\tau^* = |\tau^*_1|^{n-1}\tau^*_1. \tag{7}$$

Because of the symmetry of the results, only the lower region of the channel $-1 \leqslant y^* \leqslant 0$ is considered, where the shear stress is positive. With the transformation

$$\tau^*_1 = Re_1\tau^* = \bar{\tau}^{2/(n+1)} \tag{8}$$

the integral in (6) becomes

$$\int_{y^*}^{-1}\tau^*\mathrm{d}y^* = \frac{1}{Re_1}\int_{y^*}^{-1}\bar{\tau}^{2/(n+1)}\mathrm{d}y^* = \frac{1}{Re_1}\int_{\bar{\tau}}^{\bar{\tau}(-1)}\frac{\bar{\tau}^{2/(n+1)}\mathrm{d}\bar{\tau}}{\bar{\tau}'}. \tag{9}$$

A relation between the shear stress and its derivative can be derived from (4) and (7). Differentiating (4) and substituting the result in (7) gives

$$\tau^{*\prime\prime}_1 - Ha_1^2|\tau^*_1|^{n-1}\tau^*_1 = 0. \tag{10}$$

The integration of (10) gives

$$\tau^{*\prime}_1 = \pm\sqrt{\left[\frac{2Ha_1^2}{n+1}|\tau^*_1|^{n+1} + \tau^{*\prime 2}_1(0)\right]}. \tag{11}$$

Using the transformation (8), the integral (9) can be written as

$$-\int_{y^*}^{-1}\tau^*\mathrm{d}y^* = \frac{\sqrt{[1/(n+1)]}}{Ha_1 Re_1}\int_{\bar{\tau}}^{\bar{\tau}(-1)}\frac{\bar{\tau}^{(3-n)/(1+n)}\mathrm{d}\bar{\tau}}{\sqrt{(\bar{\tau}^2 + \beta^2)}}, \tag{12}$$

where β is given as

$$\beta = \frac{\sqrt{[(n+1)/2]}}{Ha_1}\tau^{*\prime}_1(0). \tag{13}$$

The solution of (12) is given in [12].

$$-\int_{y^*}^{-1}\tau^*_1\mathrm{d}y^* = \frac{J_{(1)} - J_{(2)}}{2\tau^{*\prime}_1(0)} \tag{14}$$

with

$$J_{(1)} = \tau^{*2}_1(-1) \times {}_3F_2\left(\frac{1}{2}, \frac{2}{1+n}, \frac{5+n}{2(1+n)}, \frac{5+n}{2(1+n)}, \frac{3+n}{1+n}, -\frac{\bar{\tau}^{2/(n+1)}}{\beta^2}\right) \tag{15}$$

$$J_{(2)} = \tau^{*2}_1 \times {}_3F_2\left(\frac{1}{2}, \frac{2}{1+n}, \frac{5+n}{2(1+n)}, \frac{5+n}{2(1+n)}, \frac{3+n}{1+n}, -\frac{\tau^{*n+1}_1}{\beta^2}\right) \tag{16}$$

where ${}_3F_2$ is the generalized second order hypergeometric function (Clausen function). The solution for the shear stress distribution can be obtained from (11) in the same manner.

$$\tau^*_1 \times {}_3F_2\left(\frac{1}{2}, \frac{1}{1+n}, \frac{3+n}{2(1+n)}, \frac{3+n}{2(1+n)}, \frac{2+n}{1+n}, -\frac{\tau^{*n+1}_1(-1)}{\beta^2}\right) = y^*. \tag{17}$$

For MHD channel flow of power law fluids, the solution for the temperature distribution can be written as

$$\theta^*_\infty = \frac{Pr_1}{2}\left[\frac{\tau^{*2}_1(-1) - \tau^{*2}_1}{Ha_1^2} + Ha_1^2(K - K^*)^2(1 - y^{*2}) + \frac{2(K - K^*)}{\tau^{*\prime}_1(0)}(J_{(1)} - J_{(2)})\right]. \tag{18}$$

Using the results of (15)–(17), the temperature distribution can easily be calculated.

MHD CHANNEL FLOW OF PRANDTL-EYRING FLUIDS

For MHD channel flow of Prandtl–Eyring fluids, the stress–strain relation and the solution of the energy equation can be written in dimensionless form as follows:

$$u^{*\prime} = Kz \sinh(2\tau_2^*), \tag{19}$$

$$\theta_\infty^* = 2Pr_2\left[\frac{Kz^2}{Ha_2^2}(\tau_2^{*2}(1) - \tau_2^{*2}) + \frac{Ha_2^2}{4}(K - K^*)^2 - 2Kz(K - K^*)\int_{y^*}^{1}\tau_2^* \mathrm{d}y^*\right], \tag{20}$$

where the dimensionless shear stress is given as $\tau_2^* = \tau/2B$. The integral in (20) can be written as

$$\int_{y^*}^{1} \tau_2^* \mathrm{d}y^* = \tau_2^*(1) - \tau_2^* y^* - \int_{\tau_2^*}^{\tau_2^*(1)} y^* \mathrm{d}\tau_2^*. \tag{21}$$

Solutions for the shear stress distributions in terms of incomplete elliptic integrals of the first kind are given in [10] as follows

$$Ha_2 \leqslant |\tau_2^{*\prime}(0)|: \quad F(\varphi_{(1)}, k_{(1)}) = \tau_2^{*\prime}(0)y^*, \tag{22}$$

$$Ha_2 \geqslant |\tau_2^{*\prime}(0)|: \quad F(\varphi_{(2)}, k_{(2)}) = F\left(\frac{\pi}{2}, k_{(2)}\right) + Ha_2 y^*. \tag{23}$$

With these solutions, the integral in (21) becomes

$$Ha_2 \leqslant |\tau_2^{*\prime}(0)|: \quad \int_{\tau_2^*}^{\tau_2^*(1)} y^* \mathrm{d}\tau_2^* = \frac{1}{\tau_2^{*\prime}(0)}\int_{\varphi_{(1)}}^{\varphi_{(1)}(1)} \frac{F(\varphi_{(1)}, k_{(1)})}{\cos\varphi_{(1)}}\mathrm{d}\varphi_{(1)}, \tag{24}$$

$$Ha_2 \geqslant |\tau_2^{*\prime}(0)|: \quad \int_{\tau_2^*}^{\tau_2^*(1)} y^* \mathrm{d}\tau_2^* = -\frac{F(\pi/2, k_{(2)})(\tau_2^*(1) - \tau_2^*)}{Ha_2} + \frac{1}{Ha_2}\int_{\varphi_{(2)}}^{\varphi_{(2)}(1)} \frac{F(\varphi_{(2)}, k_{(2)})}{\sin\varphi_{(2)}}\mathrm{d}\varphi_{(2)}. \tag{25}$$

The integrals (24) and (25) can be evaluated numerically or with help of series expansions. Using the results of (24) and (25) and the shear stress distributions given in [10], one can easily calculate the temperature distributions for the thermally fully developed region.

CONCLUSION

Considering plane laminar channel flow of heat-generating fluids, we have seen that the solution of the energy equation for the thermally fully developed region is one part of the general solution for the thermal entrance region. It has been shown that for MHD channel flow of Newtonian as well as non-Newtonian fluids, the temperature in the thermally fully developed region can be expressed in terms of the shear stress and the distance across channel height. Instead of solving the energy equation numerically for this case, only an integral of the shear stress has to be evaluated. As examples, MHD channel flow of power law fluids and Prandtl–Eyring fluids were investigated.

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REFERENCES

1. S. D. Nigam and S. N. Singh, Heat transfer by laminar flow between parallel plates under the action of transverse magnetic field, *Q. J. Mech. Appl. Math.* **13**, 85–97 (1960).
2. M. Perlmutter and R. Siegel, Heat transfer to an electrically conducting fluid flow in a channel with a transverse magnetic field, NASA Techn. Note D-875 (1961).
3. I. Michiyoshi and R. Matsumoto, Heat transfer by Hartmann's flow in thermal entrance region, *Int. J. Heat Mass Transfer* **7**, 101–112 (1964).
4. C. L. Hwang, P. J. Knieper and L. T. Fan, Heat transfer to MHD flow in the thermal entrance of a flat duct, *Int. J. Heat Mass Transfer* **9**, 773–789 (1966).
5. L. H. Back, Laminar heat transfer in electrically conducting fluids flowing in parallel plate channels, *Int. J. Heat Mass Transfer* **11**, 1621–1636 (1968).
6. R. C. Lecroy and A. H. Eraslan, The solution of temperature development in the entrance region of an MHD channel by the B. G. Galerkin method, *J. Heat Transfer* **91**, 212–220 (1969).
7. A. H. Eraslan and N. F. Eraslan, Heat transfer in magnetohydrodynamic channel flow, *Physics Fluids* **12** (1), 120–128 (1969).
8. G. W. Sutton and A. Sherman, *Engineering Magnetohydrodynamics*. McGraw-Hill, New York (1965).
9. L. K. Martinson and K. B. Pavlov, The effect of magnetic plasticity in non-Newtonian fluids, *Magnitnaya Gidrodinamika* **2** (3), 69–75 (1966).
10. P. Schroeder, Die Spaltströmung eines elektrisch leitenden Prandtl–Eyring-Fluids in einem Magnetfeld und einem elektrischen Feld, *Acta Mechanica* (to be published).
11. W. Y. Yang and Y. W. Ou, Magnetohydrodynamic channel flow of suspension, *Japan. J. Appl. Physics* **9**, 992–996 (1970).
12. I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Sums, Series and Products*. Academic Press, New York (1965).